

ABSOLUTELY SUMMING MULTILINEAR OPERATORS: A PANORAMA

DANIEL PELLEGRINO AND JOEDSON SANTOS

ABSTRACT. This paper has a twofold purpose: to present an overview of the theory of absolutely summing operators and its different generalizations for the multilinear setting, and to sketch the beginning of a research project related to an objective search of “perfect” multilinear extensions of the ideal of absolutely summing operators. The final section contains some open problems that may indicate lines for future investigation.

1. INTRODUCTION

Absolutely summing multilinear operators and homogeneous polynomials between Banach spaces were first conceived by A. Pietsch [81, 82] in the eighties. Pietsch’s work and R. Alencar and M.C. Matos’ research report [1] are usually quoted as the precursors of the now well-known nonlinear theory of absolutely summing operators. In the last decade this topic of investigation attracted the attention of many authors and various different concepts related to summability of nonlinear operators were introduced; this line of research, besides its intrinsic interests, highlighted abstract questions in the mainstream of the theory of multi-ideals which contributed to the revitalization of the general interest in questions related to ideals of polynomials and multilinear operators (see [9, 10, 15, 27, 28]).

This paper has a twofold purpose: to summarize/organize some information constructed in the last years concerning the different multilinear generalizations of absolutely summing operators; and to sketch a research project directed to the investigation of the existence of multilinear ideals (related to the ideal of absolutely summing operators) satisfying a list of properties which we consider natural. We define the notion of maximal and minimal ideals satisfying some given properties and obtain existence results, using Zorn’s Lemma. We also discuss qualitative results, posing some question on the concrete nature of the maximal and minimal ideals.

None of our goals has the intention to be exhaustive: the overview of the multilinear theory of absolutely summing operators will be concentrated in special properties and has no encyclopedic character. Besides, our approach to the existence of multi-ideals satisfying some given properties is, of course, focused on those selected properties.

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2. ABSOLUTELY SUMMING OPERATORS: AN OVERVIEW

A. Dvoretzky and C. A. Rogers [41], in 1950, solved a long standing problem in Banach Space Theory, by showing that in every infinite-dimensional Banach space there is an unconditionally convergent series which fails to be absolutely convergent. This result answers Problem 122 of the Scottish Book [60] (the problem was raised by S. Banach in [3, page 40]).

This result attracted the interest of A. Grothendieck who, in [45], presented a different proof of Dvoretzky-Rogers Theorem. Grothendieck's "Résumé de la théorie métrique des produits tensoriels topologiques" together with his thesis may be regarded, in some sense, as the birthplace of the theory of operators ideals.

The concept of absolutely p -summing linear operators is due to A. Pietsch [80] and the notion of (q, p) -summing operator is due to B. Mitiagin and A. Pełczyński [65]. Another cornerstone in the theory is J. Lindenstrauss and A. Pełczyński's paper [54], which translated Grothendieck's ideas to an universal language and showed the intrinsic beauty of the theory and richness of possible applications.

From now on the space of all continuous linear operators from a Banach space E to a Banach space F will be denoted by $\mathcal{L}(E, F)$. Let

$$\ell_p^{\text{weak}}(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \|(x_j)_{j=1}^\infty\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^\infty |\varphi(x_j)|^p \right)^{1/p} < \infty \right\}$$

and

$$\ell_p(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \|(x_j)_{j=1}^\infty\|_p := \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{1/p} < \infty \right\}.$$

If $1 \leq p \leq q < \infty$, we say that a continuous linear operator $u : E \rightarrow F$ is (q, p) -summing if $(u(x_j))_{j=1}^\infty \in \ell_q(F)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^{\text{weak}}(E)$.

The class of absolutely (q, p) -summing linear operators from E to F will be represented by $\Pi_{q,p}(E, F)$ and $\Pi_p(E, F)$ if $p = q$ (in this case $u \in \Pi_p(E, F)$ is said to be absolutely p -summing).

An equivalent formulation asserts that $u : E \rightarrow F$ is (q, p) -summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^\infty \|u(x_j)\|^q \right)^{1/q} \leq C \|(x_j)_{j=1}^\infty\|_{w,p}$$

for all $(x_j)_{j=1}^\infty \in \ell_p^{\text{weak}}(E)$. The above inequality can also be replaced by: there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^m \|u(x_j)\|^q \right)^{1/q} \leq C \|(x_j)_{j=1}^m\|_{w,p}$$

for all $x_1, \dots, x_m \in E$ and all positive integers m .

The infimum of all C that satisfy the above inequalities defines a norm, denoted by $\pi_{q,p}(u)$ (or $\pi_p(u)$ if $p = q$), and $(\Pi_{q,p}(E, F), \pi_{q,p})$ is a Banach space.

From now on, if $1 < p < \infty$, the conjugate of p is denoted by p^* , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

For a full panorama of the linear theory of absolutely summing operators we refer to the classical book [37]. Here we restrict ourselves to five pillars of the theory: Dvoretzky-Rogers Theorem, Grothendieck's Inequality, Grothendieck (Lindenstrauss-Pełczyński) ℓ_1 - ℓ_2 Theorem, Lindenstrauss-Pełczyński Theorem (on the converse of Grothendieck ℓ_1 - ℓ_2 Theorem) and Pietsch Domination Theorem.

Dvoretzky-Rogers Theorem can be stated in the context of absolutely summing operators as follows:

Theorem 2.1 (Dvoretzky-Rogers Theorem, 1950). *If $p \geq 1$, then $\Pi_p(E; E) = \mathcal{L}(E; E)$ if and only if $\dim E < \infty$.*

In view of the above result it is natural to ask for the existence of some p and infinite-dimensional Banach spaces E and F for which $\Pi_p(E; F) = \mathcal{L}(E; F)$. This question will be answered by Theorem 2.3 and Theorem 2.4 below.

The fundamental tool of the theory is Grothendieck's Inequality (the formulation below is due to Lindenstrauss and Pełczyński [54]). We omit the proof, but several different proofs can be easily found in the literature:

Theorem 2.2 (Grothendieck's Inequality (version of Lindenstrauss and Pełczyński), 1968). *There is a positive constant K_G so that, for all Hilbert space H , all $n \in \mathbb{N}$, every matrix $(a_{ij})_{n \times n}$ and any $x_1, \dots, x_n, y_1, \dots, y_n$ in the unit ball of H , the following inequality holds:*

$$(2.1) \quad \left| \sum_{i,j=1}^n a_{ij} \langle x_i, y_j \rangle \right| \leq K_G \sup \left\{ \left| \sum_{i,j=1}^n a_{ij} s_i t_j \right| : |s_i|, |t_j| \leq 1 \right\}.$$

A consequence of Grothendieck's Inequality is that every continuous linear operator from ℓ_1 to ℓ_2 is absolutely 1-summing. This result was stated by Lindenstrauss-Pełczyński [54] and is, in some sense, contained in Grothendieck's Résumé. This type of result is what is now referred to as a "coincidence theorem", i.e., a situation where there are Banach spaces E and F and real numbers $1 \leq p, q < \infty$ so that

$$\Pi_{q,p}(E, F) = \mathcal{L}(E, F).$$

The same terminology will be used for multilinear mappings.

We sketch here one of the most elementary proofs of Grothendieck (Lindenstrauss-Pełczyński) Theorem; the crucial role played by Grothendieck Inequality is easily seen.

Theorem 2.3 (Grothendieck's ℓ_1 - ℓ_2 Theorem (version of Lindenstrauss, Pełczyński), 1968). *Every continuous linear operator from ℓ_1 to ℓ_2 is absolutely 1-summing.*

Proof. Let $(T_n)_{n=1}^\infty$ be the sequence of the canonical projections, i.e.,

$$T_n : \ell_1 \longrightarrow \ell_1$$

$$x = \sum_{i=1}^{\infty} a_i e_i \mapsto T_n(x) = \sum_{i=1}^n a_i e_i.$$

Let $(x_k)_{k=1}^\infty \in \ell_1^{\text{weak}}(\ell_1)$ with

$$\|(x_k)_{k=1}^\infty\|_{w,1} = \sup_{\varphi \in B_{\ell_1^*}} \sum_{k=1}^\infty |\varphi(x_k)| \leq 1.$$

One can easily verify that

$$\|(T_n x_k)_{k=1}^\infty\|_{w,1} = \sup_{\varphi \in B_{\ell_1^*}} \sum_{k=1}^\infty |\varphi(T_n x_k)| \leq 1.$$

Denoting

$$x_k = \sum_{j=1}^\infty a_{jk} e_j \text{ and } T_n x_k = \sum_{j=1}^n a_{jk} e_j,$$

for each n, k , we can verify that for any positive integers $m \leq n$ and $(s_j)_{j=1}^n, (t_k)_{k=1}^m \subset B_{\mathbb{K}}$, we have

$$\left| \sum_{j=1}^n \sum_{k=1}^m a_{jk} s_j t_k \right| \leq 1.$$

Now, let $T \in \mathcal{L}(\ell_1, \ell_2)$ and $m, n \in \mathbb{N}$, with $n \geq m$. For each k , $1 \leq k \leq m$, from Hahn-Banach Theorem and Riesz Representation Theorem there is a $y_k \in \ell_2$, with $\|y_k\|_2 \leq 1$, so that

$$\|T T_n x_k\|_2 = \langle T T_n x_k, y_k \rangle.$$

If $m < n$, we can choose $y_{m+1} = \dots = y_n = 0$. Hence

$$\sum_{k=1}^m \|T T_n x_k\|_2 = \left| \sum_{k=1}^m \sum_{j=1}^n a_{jk} \langle T e_j, y_k \rangle \right|.$$

Now Grothendieck's Inequality comes into play:

$$(2.2) \quad \sum_{k=1}^m \|T T_n x_k\|_2 \leq K_G \|T\| \sup \left\{ \left| \sum_{j=1}^n \sum_{k=1}^m a_{jk} s_j t_k \right| : |s_j|, |t_k| \leq 1 \right\} \leq K_G \|T\|$$

for all n, m , with $n \geq m$. Since

$$\lim_{n \rightarrow \infty} T_n x_k = x_k,$$

making $n \rightarrow \infty$ in (2.2), we have

$$\sum_{k=1}^m \|T x_k\|_2 \leq K_G \|T\|$$

and the proof is done. \square

The next result is a kind of reciprocal of the Grothendieck Theorem (for a proof we refer to [54]):

Theorem 2.4 (Lindenstrauss, Pełczyński, 1968). *If E and F are infinite-dimensional Banach spaces, E has an unconditional Schauder basis and $\Pi_1(E, F) = \mathcal{L}(E, F)$ then $E = \ell_1$ and F is a Hilbert space.*

Another interesting feature of absolutely summing operators is the Domination-Theorem:

Theorem 2.5 (Pietsch-Domination Theorem, 1967). *If E and F are Banach spaces, a continuous linear operator $T : E \rightarrow F$ is absolutely p -summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on the closed unit ball of the dual of E , $(B_{E^*}, \sigma(E^*, E))$, such that*

$$(2.3) \quad \|T(x)\| \leq C \left(\int_{B_{E^*}} |\varphi(x)|^p d\mu \right)^{\frac{1}{p}}.$$

Proof. (Sketch) If (2.3) holds it is easy to show that T is absolutely p -summing. For the converse, consider the (compact) set $P(B_{E^*})$ of the probability measures in $C(B_{E^*})^*$ (endowed with the weak-star topology). For each $(x_j)_{j=1}^m$ in E , and $m \in \mathbb{N}$, let $g : P(B_{E^*}) \rightarrow \mathbb{R}$ be defined by

$$g(\rho) = \sum_{j=1}^m \left[\|T(x_j)\|^p - C^p \int_{B_{E^*}} |\varphi(x_j)|^p d\rho(\varphi) \right]$$

and \mathcal{F} be the set of all such g . It is not difficult to show that \mathcal{F} is concave and each $g \in \mathcal{F}$ is continuous and convex.

Besides, for each $g \in \mathcal{F}$ there is a measure $\mu_g \in P(B_{E^*})$ such that $g(\mu_g) \leq 0$. In fact, from the compactness of B_{E^*} and Weierstrass' theorem there is a $\varphi_0 \in K$ so that

$$\sum_{j=1}^m |\varphi_0(x_j)|^p = \sup_{\varphi \in B_{E^*}} \sum_{j=1}^m |\varphi(x_j)|^p.$$

Then, considering the Dirac measure $\mu_g = \delta_{\varphi_0}$, we deduce $g(\mu_g) \leq 0$. So, Ky Fan Lemma (see [79, page 40]) ensures that there exists a $\mu \in P(B_{E^*})$ so that

$$g(\mu) \leq 0$$

for all $g \in \mathcal{F}$ and by choosing an arbitrary g with $m = 1$ the proof is done. \square

Using the canonical inclusions from L_p spaces we get the following result:

Corollary 2.6 (Inclusion Theorem). *If $1 \leq r \leq s < \infty$, then every absolutely r -summing operator is absolutely s -summing.*

The 70's witnessed the emergence of the notion of cotype of a Banach space, with contributions from J. Hoffmann-Jørgensen [46], B. Maurey [61], S. Kwapien [53], E. Dubinsky, A. Pełczyński and H. P. Rosenthal [40] among others; in 1976 the strong connection between the notions of cotype and absolutely summing operators became evident with the work of B. Maurey and G. Pisier [62]. Let us recall the notion of cotype.

The Rademacher functions

$$r_n : [0, 1] \longrightarrow \mathbb{R}, n \in \mathbb{N}$$

are defined as

$$r_n(t) := \text{sign}(\sin 2^n \pi t).$$

A Banach space E is said to have cotype $q \geq 2$ if there is a constant $K \geq 0$ so that, for all positive integer n and all x_1, \dots, x_n in E , we have

$$(2.4) \quad \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq K \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2}.$$

We denote by $C_q(E)$ the infimum of all such K satisfying (2.4) and $\cot E$ denotes the infimum of the cotypes assumed by E , i.e.,

$$\cot E = \inf \{2 \leq q \leq \infty; E \text{ has cotype } q\}.$$

It is worth mentioning that E need not to have cotype $\cot E$.

The following combination of results of Maurey, Pisier [62] and Talagrand [90] are self-explanatory:

Theorem 2.7 (Maurey, Pisier, 1976 + Talagrand, 1992). *If a Banach space E has finite cotype q , then id_E is absolutely $(q, 1)$ -summing. The converse is true, except for $q = 2$.*

Proof. (Easy part) If E has cotype $q < \infty$, then

$$\begin{aligned} \left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} &\leq C_q(E) \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \\ &\leq C_q(E) \sup_{|t| \leq 1} \left\| \sum_{j=1}^n r_j(t) x_j \right\| \\ &\leq C_q(E) \|(x_j)_{j=1}^n\|_{w,1}. \end{aligned}$$

The rest of the proof is quite delicate. □

In the 80's the part of the focus of the investigation related to absolutely summing operators was naturally moved to the nonlinear setting, which will be treated in the next sections. However the linear theory is still alive and there are still interesting problems being investigated (see, for example, [32, 50]). For recent results we mention [21, 25, 51, 52]:

Recent results reinforce the important role played by cotype:

Theorem 2.8 (Botelho, Pellegrino, 2009). ([21, 25]) *Let E and F be infinite-dimensional Banach spaces.*

- (i) *If $\Pi_1(E, F) = \mathcal{L}(E, F)$ then $\cot E = \cot F = 2$.*
- (ii) *If $2 \leq r < \cot F$ and $\Pi_{q,r}(E, F) = \mathcal{L}(E, F)$, then $\mathcal{L}(\ell_1, \ell_{\cot F}) = \Pi_{q,r}(\ell_1, \ell_{\cot F})$.*
- (iii) *If $\cot F = \infty$ and $p \geq 1$, there exists a continuous linear operator from E to F which fails to be p -summing.*

In a completely different direction, recent papers have investigated linear absolutely summing operators in the context of the theory of lineability/spaceability (see [12, 85, 49]). For example, in [49] the following result (which can be interpreted as a generalization of results from [30]) is shown:

Theorem 2.9 (Kitson, Timoney, 2010). *Let $\mathcal{K}(E, F)$ denote the space of compact linear operators from E to F . If E and F are infinite-dimensional Banach spaces and E is super-reflexive, then*

$$A = \mathcal{K}(E, F) \setminus \bigcup_{1 \leq p < \infty} \Pi_p(E, F)$$

is spaceable (i.e., $A \cup \{0\}$ contains a closed infinite-dimensional vector space).

3. OPERATOR IDEALS AND MULTI-IDEALS: GENERATING MULTI-IDEALS

The theory of operator ideals is due to Pietsch and goes back to his monograph [79] in 1978. An operator ideal \mathcal{I} is a subclass of the class \mathcal{L} of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E; F) := \mathcal{L}(E; F) \cap \mathcal{I}$$

satisfy:

- (1) $\mathcal{I}(E; F)$ is a linear subspace of $\mathcal{L}(E; F)$ which contains the finite rank operators.
- (2) (Ideal property) If $u \in \mathcal{I}(E; F)$, $v \in \mathcal{L}(G; E)$ for $j = 1, \dots, n$ and $t \in \mathcal{L}(F; H)$, then $t \circ u \circ v \in \mathcal{I}(G; H)$.

The structure of operator ideals is shared by the most important classes of operators that appear in Functional Analysis, such as compact, weakly compact, nuclear, approximable, absolutely summing, strictly singular operators, among many others.

The multilinear theory of operator ideals was also sketched by Pietsch in [81].

From now on \mathbb{K} represents the field of all scalars (complex or real), and \mathbb{N} denotes the set of all positive integers. For $n \geq 1$, the Banach space of all continuous n -linear mappings from $E_1 \times \dots \times E_n$ into F endowed with the sup norm is denoted by $\mathcal{L}(E_1, \dots, E_n; F)$.

An ideal of multilinear mappings (or multi-ideal) \mathcal{M} is a subclass of the class of all continuous multilinear operators between Banach spaces such that for a positive integer n , Banach spaces E_1, \dots, E_n and F , the components

$$\mathcal{M}(E_1, \dots, E_n; F) := \mathcal{L}(E_1, \dots, E_n; F) \cap \mathcal{M}$$

satisfy:

- (i) $\mathcal{M}(E_1, \dots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ which contains the n -linear mappings of finite type.

- (ii) The ideal property: if $A \in \mathcal{M}(E_1, \dots, E_n; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, n$ and $t \in \mathcal{L}(F; H)$, then $t \circ A \circ (u_1, \dots, u_n)$ belongs to $\mathcal{M}(G_1, \dots, G_n; H)$.

Moreover, there is a function $\|\cdot\|_{\mathcal{M}}: \mathcal{M} \rightarrow [0, \infty)$ satisfying

- (i') $\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}(E_1, \dots, E_n; F)$ is a norm, for all Banach spaces E_1, \dots, E_n and F , which makes $\mathcal{M}(E_1, \dots, E_n; F)$ a Banach space.

- (ii') $\|A: \mathbb{K}^n \rightarrow \mathbb{K}: A(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{M}} = 1$ for all n ,

- (iii') If $A \in \mathcal{M}(E_1, \dots, E_n; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, n$ and $v \in \mathcal{L}(F; H)$, then $\|v \circ A \circ (u_1, \dots, u_n)\|_{\mathcal{M}} \leq \|v\| \|A\|_{\mathcal{M}} \|u_1\| \cdots \|u_n\|$.

However, the construction of adequate multilinear and polynomial extensions of a given operator ideal needs some care. The first is that, given positive integers n_1

and n_2 , the respective levels of n_1 -linearity and n_2 -linearity need to have some inter-connection and obviously a strong relation with the original level ($n = 1$). This pertinent preoccupation has appeared in different recent papers, with the notions of ideals closed for scalar multiplication, closed for differentiation and the notions of coherent and compatible multilinear ideals (see [10, 15, 27]).

The following properties illustrate the essence of the aforementioned inter-connection between the levels of the multi-ideal (these concepts are natural adaptations from the analogous for polynomials defined in [15]).

Definition 3.1 (cud multi-ideal). *An ideal of multilinear mappings \mathcal{M} is closed under differentiation (cud) if, for all n , E_1, \dots, E_n, F and $T \in \mathcal{M}(E_1, \dots, E_n; F)$, every linear operator obtained by fixing $n - 1$ vectors $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ belongs to $\mathcal{M}(E_j; F)$ for all $j = 1, \dots, n$.*

Definition 3.2 (csm multi-ideal). *An ideal of multilinear mappings \mathcal{M} is closed for scalar multiplication (csm) if for all n , $E_1, \dots, E_n, E_{n+1}, F$, $T \in \mathcal{M}(E_1, \dots, E_n; F)$ and $\varphi \in E_{n+1}^*$, the map φT belongs to $\mathcal{M}(E_1, \dots, E_n, E_{n+1}; F)$.*

For the theory of polynomials and multilinear mappings between Banach spaces we refer to [39, 66].

4. MULTIPLE SUMMING MULTILINEAR OPERATORS: THE PRIZED IDEA

Few know that the concept of multiple p -summing mappings was introduced in a research report of M.C. Matos in 1992 [56], under the terminology of “strictly absolutely summing multilinear mappings”. The motivation of Matos was a question of Pietsch on the eventual coincidence of the Hilbert-Schmidt n -linear functionals and the space of absolutely $(s; r_1, \dots, r_n)$ -summing n -linear functionals for some values of s and $r_k, k = 1, \dots, n$. In this research report, the first properties of this class are introduced, as well as the connections with Hilbert-Schmidt multilinear operators and a solution to Pietsch’s question in the context of strictly absolutely summing multilinear mappings.

However, this research report was not published and only in 2003 Matos [58] published an improved version of this preprint, now using the terminology of *fully summing multilinear mappings*. At the same time, and independently, Bombal, Pérez-García and Villanueva [7, 77] introduced the same concept, under the terminology of multiple summing multilinear operators.

Since then this class has gained special attention, being considered by several authors as the most important multilinear generalization of the ideal of absolutely summing operators. For this reason we will dedicate more attention to the description of this class.

A fair description of the subject should begin in 1930, when Littlewood [55] (see [4] for a recent approach) proved his Littlewood’s $4/3$ inequality asserting that

$$\left(\sum_{i,j=1}^N |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|U\|$$

for every bilinear form $U : \ell_\infty^N \times \ell_\infty^N \rightarrow \mathbb{C}$ and every positive integer N . One year later Bohnenblust and Hille [6] (see also [34, 35]) improved this result to multilinear forms by showing that for every positive integer n there is a $C_n > 0$ so that

$$(4.1) \quad \left(\sum_{i_1, \dots, i_n=1}^N |U(e_{i_1}, \dots, e_{i_n})|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq C_n \|U\|$$

for every n -linear mapping $U : \ell_\infty^N \times \dots \times \ell_\infty^N \rightarrow \mathbb{C}$ and every positive integer N .

Using that $\mathcal{L}(c_0; E)$ is isometrically isomorphic to $\ell_1^w(E)$ (see [37]), Bohnenblust-Hille inequality can be re-written as (details can be found in [73]):

Theorem 4.1 (Bohnenblust-Hille, re-written (Pérez-García, 2003)). *If $1 \leq p < \infty$, and n is a positive integer and E_1, \dots, E_n are Banach spaces and $U \in \mathcal{L}(E_1, \dots, E_n; \mathbb{K})$, then there exists a constant $C_n \geq 0$ such that*

$$(4.2) \quad \left(\sum_{j_1, \dots, j_n=1}^N \left| U(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)}) \right|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq C_n \prod_{k=1}^n \left\| (x_j^{(k)})_{j=1}^N \right\|_{w,1}$$

for every positive integer N and $x_j^{(k)} \in E_k$, $k = 1, \dots, n$ and $j = 1, \dots, N$.

In this sense Bohnenblust-Hille theorem can be interpreted as the beginning of the notion of multiple summing operators:

If $1 \leq p_1, \dots, p_n \leq q < \infty$, $T : E_1 \times \dots \times E_n \rightarrow F$ is multiple $(q; p_1, \dots, p_n)$ -summing ($T \in \mathcal{L}_{m,(q,p_1,\dots,p_n)}(E_1, \dots, E_n; F)$) if there exists $C_n > 0$ such that

$$(4.3) \quad \left(\sum_{j_1, \dots, j_n=1}^\infty \|T(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)})\|^q \right)^{1/q} \leq C_n \prod_{k=1}^n \|(x_j^{(k)})_{j=1}^\infty\|_{w,p_k}$$

for every $(x_j^{(k)})_{j=1}^\infty \in \ell_{p_k}^w(E_k)$, $k = 1, \dots, n$.

When $p_1 = \dots = p_n = p$ we write $\mathcal{L}_{m,(q,p)}$ instead of $\mathcal{L}_{m,(q;p_1,\dots,p_n)}$; when $p_1 = \dots = p_n = p = q$ we write $\mathcal{L}_{m,p}$ instead of $\mathcal{L}_{m,(q;p_1,\dots,p_n)}$. The infimum of the constants C_n satisfying (4.3) defines a norm in $\mathcal{L}_{m,(q,p)}$ and is denoted by $\pi_{q;p_1,\dots,p_k}$ (or $\pi_{q;p}$ if $p_1 = \dots = p_k = p$ or even π_p when $p_1 = \dots = p_k = p = q$). It is worth mentioning that the essence of the notion of multiple summing multilinear operators, for bilinear operators, also appears in the paper of Ramanujan and Schock [87].

It is well-known that the power $\frac{2n}{n+1}$ in Bohnenblust-Hille Theorem 4.1 is optimal. The constant C_n from (4.2) is the same constant from (4.1). The optimal values are not known. For recent estimates for C_n we refer to [70]. For example, in the real case, for $2 \leq n \leq 14$, in [70] it is shown that $C_n \leq 2^{\frac{n^2+6n-8}{8n}}$ if n is even and by $C_n \leq 2^{\frac{n^2+6n-7}{8n}}$ if n is odd (these estimates are derived from [35]). In the complex case, H. Quéffelec [86], A. Defant and P. Sevilla-Peris [34] have proved that $C_n \leq \left(\frac{2}{\sqrt{\pi}}\right)^{n-1}$ but for $n \geq 8$ better estimates can be also found in [70] (also derived from [35]).

So, since the power $\frac{2n}{n+1}$ is sharp, one might not expect that the class of multiple summing operators shall lift the trivial coincidence situations from the linear case, i.e.,

$$\Pi_p(E; \mathbb{K}) = \mathcal{L}(E, \mathbb{K})$$

for every Banach spaces E but, in general,

$$\mathcal{L}_{m,p}(^n E; \mathbb{K}) \neq \mathcal{L}(^n E, \mathbb{K}).$$

The multi-ideal of multiple summing multilinear operators is, by far, the most investigated class related to the multilinear theory of absolutely summing operators (see [20, 33, 34, 74] and references therein). The reason for the success of this generalization of absolutely summing operators is perhaps the nice combination of nontrivial good properties, as coincidence theorems similar to those from the linear theory ([7, 11, 20]), and challenging problems as the inclusion theorem which holds in very special situations.

The main results below are presented with the respective dates. In the case of results that appeared in a thesis or dissertation and were published after, we have chosen the date of the thesis/dissertation.

A first remark on the class of multiple summing multilinear operators is that it is easy to show that coincidence results for multiple summing multilinear operators always imply in the respective linear ones (details can be found in [71]):

Proposition 4.2. *If $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m,(q;p_1, \dots, p_n)}(E_1, \dots, E_n; F)$, then*

$$\mathcal{L}(E_j; F) = \Pi_{q,p_j}(E_j; F), j = 1, \dots, n.$$

Bohnenblust-Hille type results were also studied in a different perspective (trying to replace $\frac{2n}{n+1}$ by 2 by changing the 1-weak norm by some p_n -weak norm). If $(p_k)_{k=0}^\infty$ is the sequence of real numbers given by

$$p_0 = 2 \text{ and } p_{k+1} = \frac{2p_k}{1 + p_k} \text{ for } k \geq 0,$$

then the following Bohnenblust-Hille type result is valid:

Theorem 4.3 (Botelho, Braunss, Junek, Pellegrino, 2009). ([11]) *Let E_1, \dots, E_n be Banach spaces of cotype 2. If k is the natural number such that $2^{k-1} < n \leq 2^k$, then*

$$\mathcal{L}(E_1, \dots, E_n; \mathbb{K}) = \mathcal{L}_{m(2;p_k, \dots, p_k)}(E_1, \dots, E_n; \mathbb{K}).$$

A very important contribution to the theory of multiple summing multilinear operators was given in D. Pérez-García's thesis, where several new results and techniques are presented, inspiring several related papers. The inclusion theorems proved by Pérez-García deserves special attention:

Theorem 4.4 (Pérez-García, 2003). ([73, 74]) *If $1 \leq p \leq q < 2$, then $\mathcal{L}_{m,p}(E_1, \dots, E_n; F) \subset \mathcal{L}_{m,q}(E_1, \dots, E_n; F)$.*

Pérez-García has also shown that the above result cannot be extended in the sense that for each $q > 2$ there exists $T \in \mathcal{L}_{m,p}(^2 \ell_1; \mathbb{K})$ for $1 \leq p \leq 2$ which does not belong to $\mathcal{L}_{m,q}(^2 \ell_1; \mathbb{K})$.

When the F has cotype 2 the result is slight better:

Theorem 4.5 (Pérez-García, 2003). ([73, 74]) *If $1 \leq p \leq q \leq 2$ and F has cotype 2, then $\mathcal{L}_{m,p}(E_1, \dots, E_n; F) \subset \mathcal{L}_{m,q}(E_1, \dots, E_n; F)$.*

When the spaces from the domain have cotype 2, the inclusions from Theorem 4.4 become coincidences (for a simple proof we refer to [14, 19]; the main tool used in the proof are results from [2]):

Theorem 4.6 (Botelho, Pellegrino, 2008 and Popa, 2009). ([19, 84]) *If $1 \leq p, q < 2$ and E_1, \dots, E_n have cotype 2, then*

$$\mathcal{L}_{m,p}(E_1, \dots, E_n; F) = \mathcal{L}_{m,q}(E_1, \dots, E_n; F).$$

Recently, in [14], it was shown (using complex interpolation and an argument of complexification) that a more general version of Theorem 4.5 is valid when the spaces from the domain are \mathcal{L}_∞ -spaces:

Theorem 4.7 (Botelho, Michels, Pellegrino, 2010). *Let $1 \leq p \leq q \leq \infty$ and E_1, \dots, E_n be \mathcal{L}_∞ -spaces. Then $\mathcal{L}_{m,p}(E_1, \dots, E_n; F) \subset \mathcal{L}_{m,q}(E_1, \dots, E_n; F)$.*

The proofs of the above results are technical and we omit them. For other related results we mention ([11, 14, 74]).

Coincidence theorems are also a fruitful subject in the context of multiple summing operators. For example, D. Pérez-García proved that Grothendieck's Theorem is valid for multiple summing multilinear operators:

Theorem 4.8 (Pérez-García, 2003). [73] *If $1 \leq p \leq 2$, then $\mathcal{L}_{m,p}({}^n\ell_1; \ell_2) = \mathcal{L}({}^n\ell_1; \ell_2)$.*

We sketch the proof of a more general result from [20], which is inspired in Pérez-García's ideas:

Theorem 4.9 (Botelho, Pellegrino, 2009). *Let $r \geq s \geq 1$. If $\mathcal{L}(\ell_1; F) = \Pi_{r;s}(\ell_1; F)$, then*

$$\mathcal{L}({}^n\ell_1; F) = \mathcal{L}_{m,(r; \min\{s,2\})}({}^n\ell_1; F)$$

for every $n \in \mathbb{N}$.

Proof. (Sketch) In [77, Theorem 3.4] it is shown that when $1 \leq p \leq 2$, then $\mathcal{L}_{m,p}({}^2\ell_1; \mathbb{K}) = \mathcal{L}({}^2\ell_1; \mathbb{K})$, and

$$(4.4) \quad \pi_p(\cdot) \leq K_G^2 \|\cdot\|.$$

Let $(x_j^{(1)})_{j=1}^{m_1}, \dots, (x_j^{(n)})_{j=1}^{m_n}$ be n finite sequences in ℓ_1 . Using that $\widehat{\otimes}_\pi^k \ell_1$ is isometrically isomorphic to ℓ_1 and (4.4), one can prove that, for every $1 \leq p \leq 2$,

$$\left\| (x_{j_1}^{(1)} \otimes \dots \otimes x_{j_n}^{(n)})_{j_1, \dots, j_n=1}^{m_1, \dots, m_n} \right\|_{w,p} \leq K_G^{2n-2} \left\| (x_j^{(1)})_{j=1}^{m_1} \right\|_{w,p} \cdots \left\| (x_j^{(n)})_{j=1}^{m_n} \right\|_{w,p}$$

Let $A \in \mathcal{L}({}^n\ell_1; F)$. By A_L we mean the linearization of A on $\widehat{\otimes}_\pi^n \ell_1$, that is $A_L \in \mathcal{L}(\widehat{\otimes}_\pi^n \ell_1; F)$ and $A_L(x_1 \otimes \dots \otimes x_n) = A(x_1, \dots, x_n)$ for every $x_j \in \ell_1$. Since $\widehat{\otimes}_\pi^n \ell_1$ is isometrically isomorphic to ℓ_1 , by assumption we have that A_L is $(r; s)$ -summing and $\pi_{r;s}(A_L) \leq M \|A_L\| = M \|A\|$, where M is a constant independent of A . Using the claim

with $p = \min\{s, 2\}$ we get

$$\begin{aligned}
\left(\sum_{j_1, \dots, j_n=1}^{m_1, \dots, m_n} \|A(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)})\|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{j_1, \dots, j_n=1}^{m_1, \dots, m_n} \|A_L(x_{j_1}^{(1)} \otimes \dots \otimes x_{j_n}^{(n)})\|^r \right)^{\frac{1}{r}} \\
&\leq \pi_{r;s}(A_L) \left\| (x_{j_1}^{(1)} \otimes \dots \otimes x_{j_n}^{(n)})_{j_1, \dots, j_n=1}^{m_1, \dots, m_n} \right\|_{w,s} \\
&\leq \pi_{r;s}(A_L) \left\| (x_{j_1}^{(1)} \otimes \dots \otimes x_{j_n}^{(n)})_{j_1, \dots, j_n=1}^{m_1, \dots, m_n} \right\|_{w, \min\{s, 2\}} \\
&\leq M \|A\| K_G^{2n-2} \left\| (x_j^{(1)})_{j=1}^{m_1} \right\|_{w, \min\{s, 2\}} \dots \left\| (x_j^{(n)})_{j=1}^{m_n} \right\|_{w, \min\{s, 2\}},
\end{aligned}$$

which shows that A is multiple $(r; \min\{s, 2\})$ -summing. \square

Corollary 4.10 (Botelho, Pellegrino, 2009). ([20]) *Let $1 \leq p \leq 2$, $r \geq p$ and let F be a Banach space. The following assertions are equivalent:*

- (a) $\mathcal{L}(\ell_1; F) = \mathcal{L}_{m(r;p)}(^n\ell_1; F)$.
- (b) $\mathcal{L}(^n\ell_1; F) = \mathcal{L}_{m(r;p)}(^n\ell_1; F)$ for every $n \in \mathbb{N}$.
- (c) $\mathcal{L}(^n\ell_1; F) = \mathcal{L}_{m(r;p)}(^n\ell_1; F)$ for some $n \in \mathbb{N}$.

The connection between linear coincidence results with coincidence results for multiple summing multilinear operators is indeed stronger:

Theorem 4.11 (Botelho, Pellegrino, 2009). ([20]) *Let $p, r \in [1, q]$ and let F be a Banach space. Let $B(p, q, r, F)$ denote the set of all Banach spaces E such that*

$$\mathcal{L}(E; F) = \Pi_{q;p}(E; F) \text{ and } \mathcal{L}(E; \ell_q(F)) = \Pi_{q;r}(E; \ell_q(F)).$$

Then, for every $n \geq 2$,

$$\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m(q;r, \dots, r, p)}(E_1, \dots, E_n; F)$$

whenever $E_1, \dots, E_n \in B(p, q, r, F)$.

Proof. (Sketch) Induction on n . For the case $n = 2$, let $E_1, E_2 \in B(p, q, r, F)$. By the Open Mapping Theorem there are constants C_1 and C_2 such that

$$\pi_{q;p}(u) \leq C_1 \|u\| \text{ for every } u \in \mathcal{L}(E_2; F) \text{ and}$$

$$\pi_{q;r}(v) \leq C_2 \|v\| \text{ for every } v \in \mathcal{L}(E_1; \ell_q(F)).$$

Let $A \in \mathcal{L}(E_1, E_2; F)$. Given two sequences $(x_j^{(1)})_{j=1}^\infty \in \ell_r^w(E_1)$ and $(x_j^{(2)})_{j=1}^\infty \in \ell_p^w(E_2)$, fix $m \in \mathbb{N}$ and consider the continuous linear operator

$$A_1^{(m)}: E_1 \longrightarrow \ell_q(F) : A_1^{(m)}(x) = (A(x, x_1^{(2)}), \dots, A(x, x_m^{(2)}), 0, 0, \dots).$$

So, $A_1^{(m)}$ is $(q; r)$ -summing and $\pi_{q;r}(A_1^{(m)}) \leq C_2 \|A_1^{(m)}\|$. For each $x \in B_{E_1}$, consider the continuous linear operator

$$A_x: E_2 \longrightarrow F : A_x(y) = A(x, y).$$

So, A_x is $(q; p)$ -summing and $\pi_{q;p}(A_x) \leq C_1 \|A_x\| \leq C_1 \|A\| \|x\| \leq C_1 \|A\|$ and we can obtain

$$\left(\sum_{j=1}^m \sum_{k=1}^m \left\| A(x_j^{(1)}, x_k^{(2)}) \right\|^q \right)^{\frac{1}{q}} \leq C_1 C_2 \|A\| \left\| (x_j^{(1)})_{j=1}^m \right\|_{w,r} \left\| (x_k^{(2)})_{j=1}^m \right\|_{w,p},$$

which shows that A is multiple $(q; r, p)$ -summing and $\pi_{q;r,p}(A) \leq C_1 C_2 \|A\|$.

Suppose now that the result holds for n , that is: for every $E_1, \dots, E_n \in B(p, q, r, F)$, $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m(q;r,\dots,r,p)}(E_1, \dots, E_n; F)$. To prove the case $n+1$, let $E_1, \dots, E_{n+1} \in B(p, q, r, F)$. Since E_2, \dots, E_{n+1} belong to $B(p, q, r, F)$, we have $\mathcal{L}(E_2, \dots, E_{n+1}; F) = \mathcal{L}_{m(q;r,\dots,r,p)}(E_2, \dots, E_{n+1}; F)$ by the induction hypotheses and hence there is a constant C_1 such that

$$\pi_{q;r,\dots,r,p}(B) \leq C_1 \|B\| \text{ for every } B \in \mathcal{L}(E_2, \dots, E_{n+1}; F).$$

Since $E_1 \in B(p, q, r, F)$, there is a constant C_2 such that

$$\pi_{q;r}(v) \leq C_2 \|v\| \text{ for every } v \in \mathcal{L}(E_1; \ell_q(F)).$$

Let $A \in \mathcal{L}(E_1, \dots, E_{n+1}; F)$. Given sequences $(x_j^{(1)})_{j=1}^\infty \in \ell_r^w(E_1), \dots, (x_j^{(n)})_{j=1}^\infty \in \ell_r^w(E_n)$ and $(x_j^{(n+1)})_{j=1}^\infty \in \ell_p^w(E_{n+1})$, fix $m \in \mathbb{N}$ and consider the continuous linear operator

$$A_1^{(m)} : E_1 \longrightarrow \ell_q(F) : A_1^{(m)}(x) = \left((A(x, x_{j_2}^{(2)}, \dots, x_{j_{n+1}}^{(n+1)}))_{j_2, \dots, j_{n+1}=1}^m, 0, 0, \dots \right).$$

So, $A_1^{(m)}$ is $(q; r)$ -summing and $\pi_{q;r}(A_1^{(m)}) \leq C_2 \|A_1^{(m)}\|$. For each $x \in B_{E_1}$, consider the continuous n -linear mapping

$$A_x^n : E_2 \times \dots \times E_{n+1} \longrightarrow F : A_x^n(x_2, \dots, x_{n+1}) = A(x, x_2, \dots, x_{n+1}).$$

So,

$$\pi_{q;r,\dots,r,p}(A_x^n) \leq C_1 \|A_x^n\| \leq C_1 \|A\| \|x\| \leq C_1 \|A\|$$

and we conclude that

$$\left(\sum_{j_1=1}^m \dots \sum_{j_{n+1}=1}^m \left\| A(x_{j_1}^{(1)}, \dots, x_{j_{n+1}}^{(n+1)}) \right\|^q \right)^{\frac{1}{q}} \leq C_1 C_2 \|A\| \left(\prod_{k=1}^n \left\| (x_j^{(k)})_{j=1}^m \right\|_{w,r} \right) \left\| (x_j^{(n+1)})_{j=1}^m \right\|_{w,p}.$$

□

Corollary 4.12 (Souza, 2003, Pérez-García, 2003). ([7, 73, 89]) *If F has cotype q and E_1, \dots, E_n are arbitrary Banach spaces, then*

$$\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m(q;1)}(E_1, \dots, E_n; F) \text{ and}$$

$$\pi_{q;1}(A) \leq C_q(F)^n \|A\| \text{ for every } A \in \mathcal{L}(E_1, \dots, E_n; F),$$

where $C_q(F)$ is the cotype q constant of F .

Proof. Both F and $\ell_q(F)$ have cotype q (see [37, Theorem 11.12]), so $\mathcal{L}(E; F) = \Pi_{q;1}(E; F)$ and $\mathcal{L}(E; \ell_q(F)) = \Pi_{q;1}(E; \ell_q(F))$ for every Banach space E by [37, Corollary 11.17]. □

Corollary 4.13 (Pérez-García, 2003). ([7, 73]) *If E_1, \dots, E_n are \mathcal{L}_1 -spaces and H is a Hilbert space, then*

$$\mathcal{L}(E_1, \dots, E_n; H) = \mathcal{L}_{m,2}(E_1, \dots, E_n; H) \text{ and}$$

$$\pi_2(A) \leq K_G^n \|A\| \text{ for every } A \in \mathcal{L}(E_1, \dots, E_n; H),$$

where K_G stands for the Grothendieck constant.

Proof. From [31, Ex. 23.17(a)] we know that H and $\ell_2(H)$ are \mathcal{L}_2 -spaces, so $\mathcal{L}(E; H) = \Pi_{2,2}(E; H)$ and $\mathcal{L}(E; \ell_2(H)) = \Pi_{2,2}(E; \ell_2(H))$ for every \mathcal{L}_1 -space E by [37, Theorems 3.1 and 2.8]. \square

Corollary 4.14 (Pérez-García, 2003). ([7, 73]) *If F has cotype 2 and E_1, \dots, E_n are \mathcal{L}_∞ -spaces, then $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m,2}(E_1, \dots, E_n; F)$.*

Proof. From [37, Theorem 11.12] we know that F and $\ell_2(F)$ have cotype 2, so $\mathcal{L}(E; F) = \Pi_{2,2}(E; F)$ and $\mathcal{L}(E; \ell_2(F)) = \Pi_{2,2}(E; \ell_2(F))$ for every \mathcal{L}_∞ -space E by [37, Theorem 11.14(a)]. \square

By invoking [37, Theorem 11.14(b)] instead of [37, Theorem 11.14(a)] we get:

Corollary 4.15. *If F has cotype $q > 2$, E_1, \dots, E_n are \mathcal{L}_∞ -spaces and $r < q$, then $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{m(q,r)}(E_1, \dots, E_n; F)$.*

Very recently, in a remarkable paper [35], A. Defant, D. Popa and U. Schwaning introduced the notion of coordinatewise multiple summing operators and, among various interesting results, presented the following vector-valued generalization of Bohnenblust-Hille inequality. Below, a multilinear map $U \in \mathcal{L}(^n E_1, \dots, E_n; F)$ is separately $(r, 1)$ -summing if it is absolutely $(r, 1)$ -summing in each coordinate separately.

Theorem 4.16 (Defant, Popa, Schwaning, 2010). *Let F be a Banach space with cotype q , and $1 \leq r < q$. Then each separately $(r, 1)$ -summing $U \in \mathcal{L}(E_1, \dots, E_n; F)$ is multiple $(\frac{qr^n}{q+(n-1)r}, 1)$ -summing.*

Using $F = \mathbb{K}$, $q = 2$ and $r = 1$ in the above theorem, Bohnenblust-Hille Theorem is recovered.

A last comment about the richness of applications of the class of absolutely summing multilinear operators is related to tensor norms.

A. Defant and D. Pérez-García [33] constructed an n -tensor norm, in the sense of Grothendieck (associated to the class of multiple 1-summing multilinear forms) possessing the surprising property that the α -tensor product $\alpha(Y_1, \dots, Y_n)$ has local unconditional structure for each choice of n arbitrary \mathcal{L}_{p_j} -spaces Y_j . This construction answers a question posed by J. Diestel. It is interesting to mention that in [78] it is shown that none of Grothendieck's 14 norms satisfies such condition.

5. OTHER ATTEMPTS OF MULTI-IDEALS RELATED TO ABSOLUTELY SUMMING OPERATORS: AN OVERVIEW

In the last decade several classes of multilinear maps have been investigated as extensions of the linear concept of absolutely summing operators (for works comparing

these different classes we refer to [26, 76]). Depending on the properties that a given class possesses, this class is usually compared with the original linear ideal and, in some sense, qualified as a good (or bad) extension of the linear ideal. In this direction, the ideals of dominated multilinear operators and multiple summing multilinear operators are mostly classified as nice generalizations of absolutely summing linear operators. Of course, the evaluation of what properties are important or not has a subjective component, but some classical properties of absolutely summing operators are naturally expected to hold in the context of a reasonable multilinear generalization.

The usual procedure in the multilinear and polynomial theory of absolutely summing operators is to define a class and study their properties. The final sections of this paper have a different purpose; we elect some properties that we consider fundamental and investigate which classes satisfy them (specially if there exist maximal and minimal classes, in a sense that will be clear soon).

Below we sketch an overview of the different multilinear approaches to summability of operators which have arisen in the last years:

5.1. Dominated multilinear operators: the first attempt. If $p \geq 1$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be p -dominated ($T \in \mathcal{L}_{d,p}(E_1, \dots, E_n; F)$) if $(T(x_j^1, \dots, x_j^n))_{j=1}^\infty \in \ell_{p/n}(F)$ whenever $(x_j^k)_{j=1}^\infty \in \ell_p^w(E_k)$. This concept was essentially introduced by Pietsch and explored in [1, 57, 88] and has strong similarity with the original linear ideal of absolutely summing operators; during some time (before the emergence of the class of multiple summing multilinear operators) this ideal seemed to be considered as the most promising multilinear approach to summability (however, as it will be clear soon, this class is, in some sense, too small). The terminology “ p -dominated” is justified by the Pietsch-Domination type theorem (a detailed proof can be found in [72] or as a consequence of a more general result [69]):

Theorem 5.1 (Pietsch, Geiss, 1985). ([43]) *$T \in \mathcal{L}(E_1, \dots, E_n; F)$ is p -dominated if and only if there exist $C \geq 0$ and regular probability measures μ_j on the Borel σ -algebras of $B_{E_j'}$ endowed with the weak star topologies such that*

$$\|T(x_1, \dots, x_n)\| \leq C \prod_{j=1}^n \left(\int_{B_{E_j'}} |\varphi(x_j)|^p d\mu_j(\varphi) \right)^{1/p}$$

for every $x_j \in E_j$ and $j = 1, \dots, n$.

Corollary 5.2. *If $1 \leq p \leq q < \infty$, then $\mathcal{L}_{d,p} \subset \mathcal{L}_{d,q}$.*

This class has several other similarities with the linear concept of absolutely summing operators. We mention two results whose proofs mimic the linear analogues:

Theorem 5.3 (Meléndez-Tonge, 1999). ([63]) *Let $2 < p < r^* < \infty$. Let n be a positive integer and F be a Banach space. Then*

$$\mathcal{L}_{d,1}({}^n\ell_p; F) = \mathcal{L}_{d,r}({}^n\ell_p; F).$$

Theorem 5.4 (Extrapolation Theorem, 2005). ([68]) *If $1 < r < p < \infty$ and E is a Banach space such that*

$$\mathcal{L}_{d,p}(^n E; \ell_p) = \mathcal{L}_{d,r}(^n E; \ell_p),$$

then

$$\mathcal{L}_{d,p}(^n E; F) = \mathcal{L}_{d,1}(^n E; F)$$

for every Banach space F .

A consequence of Grothendieck's Inequality ensures a rare coincidence situation for this class (this result seems to be part of the folklore of the theory):

Theorem 5.5. $\mathcal{L}_{d,2}(^2 c_0; \mathbb{K}) = \mathcal{L}(^2 c_0; \mathbb{K})$.

Proof. (Real case) It suffices to deal with $A : \ell_\infty^m \times \ell_\infty^m \rightarrow \mathbb{R}$ with $\|A\| \leq 1$. Note that

$$|A(x, y)| = \left| A \left(\sum_{i=1}^m x_i e_i, \sum_{i=1}^m y_i e_i \right) \right| = \left| \sum_{i,j=1}^m A(e_i, e_j) x_i y_j \right|.$$

Let $a_{ij} = A(e_i, e_j)$ and $(x_k)_{k=1}^N, (y_k)_{k=1}^N \in \ell_2^w(\ell_\infty^m)$ be so that $\|(x_k)_{k=1}^N\|_{w,2}, \|(y_k)_{k=1}^N\|_{w,2} \leq 1$, with

$$x_k = (x_k^{(1)}, \dots, x_k^{(m)}) \text{ and } y_k = (y_k^{(1)}, \dots, y_k^{(m)}).$$

Hence, for $i, j = 1, \dots, m$, consider

$$\tilde{x}_i := (x_1^{(i)}, \dots, x_N^{(i)}) \in \ell_2^N \text{ and } \tilde{y}_j := (y_1^{(j)}, \dots, y_N^{(j)}) \in \ell_2^N.$$

It is well-known (see, for example, [72, Proposición 5.18]) that

$$\|(x_k)_{k=1}^N\|_{w,2}^2 = \max_{1 \leq i \leq m} \|\tilde{x}_i\|^2 \text{ and } \|(y_k)_{k=1}^N\|_{w,2}^2 = \max_{1 \leq j \leq m} \|\tilde{y}_j\|^2.$$

So we have $\|\tilde{x}_i\| \leq 1, \|\tilde{y}_j\| \leq 1$ for every $i, j = 1, \dots, m$, and, since $\|A\| \leq 1$, from Grothendieck's Inequality we have

$$\left| \sum_{i,j=1}^m a_{ij} \langle \tilde{x}_i, \tilde{y}_j \rangle \right| \leq K_G,$$

and therefore

$$\left| \sum_{i,j=1}^m a_{ij} \sum_{k=1}^N x_k^{(i)} y_k^{(j)} \right| \leq K_G$$

i.e.,

$$\left| \sum_{k=1}^N \left(\sum_{i,j=1}^m a_{ij} x_k^{(i)} y_k^{(j)} \right) \right| \leq K_G$$

and

$$\left| \sum_{k=1}^N A(x_k, y_k) \right| = \left| \sum_{k=1}^N A \left(\sum_{i=1}^m x_k^{(i)} e_i, \sum_{j=1}^m y_k^{(j)} e_j \right) \right| \leq K_G.$$

Since x_k can be replaced by $\varepsilon_k x_k$ with $\varepsilon_k = 1$ or -1 , we can conclude that

$$\sum_{k=1}^N |A(x_k, y_k)| \leq K_G.$$

□

In fact the result above is valid for \mathcal{L}_∞ spaces instead of c_0 . For a direct proof of this result to $C(K)$ spaces we refer to [8].

It is also known that dominated multilinear maps satisfy a Dvoretzky-Rogers type theorem ($\mathcal{L}_{d,p}({}^n E; E) = \mathcal{L}({}^n E; E)$ if and only if $\dim E < \infty$). Recent results show that this class is too small, in some sense (coincidence situations are almost impossible). The proof of the next result presented here is different from the original [47], and appears in [21]:

Theorem 5.6 (Jarchow, Palazuelos, Pérez-García and Villanueva, 2007). ([47]) *For every $n \geq 3$ and every $p \geq 1$ and every infinite dimensional Banach space E there exists $T \in \mathcal{L}({}^n E; \mathbb{K})$ that fails to be p -dominated.*

Proof. Suppose that every $T \in \mathcal{L}({}^3 E; \mathbb{K})$ is p -dominated. From [8, Lemma 3.4] one can conclude that every continuous linear operator from E to $\mathcal{L}(E; \mathcal{L}({}^2 E; \mathbb{K}))$ is p -summing. From [37, Proposition 19.17] we know that $\mathcal{L}({}^2 E; \mathbb{K})$ has no finite cotype, but from Theorem 2.8 (iii) this is not possible. Since the result is true for $n = 3$, it is easy to conclude that it is true for $n > 3$. □

For polynomial versions of this result we refer to [16, 22] and for more results on dominated multilinear operators/polynomials we refer to [8, 16, 29, 47, 63] and references therein.

Since Theorem 5.6 is valid for $n \geq 3$, a natural question is: are there coincidence situations for $n = 2$ different from the obvious variations of Theorem 5.5? The answer is yes:

Theorem 5.7 (Botelho, Pellegrino, Rueda, 2010). ([24]) *Let E be a cotype 2 space. Then $E \widehat{\otimes}_\pi E = E \widehat{\otimes}_\varepsilon E$ if and only if $\mathcal{L}_{d,1}({}^2 E; \mathbb{K}) = \mathcal{L}({}^2 E; \mathbb{K})$.*

The existence of spaces fulfilling the hypotheses of Theorem 5.7 is assured by G. Pisier [83]. Also, $\cot E = 2$ is a necessary condition for Theorem 5.7 since in [24] it is also proved that

$$\mathcal{L}_{d,1}({}^2 E; \mathbb{K}) = \mathcal{L}({}^2 E; \mathbb{K}) \Rightarrow \cot E = 2.$$

5.2. Semi-integral multilinear operators. If $p \geq 1$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is p -semi-integral ($T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$) if there exists a $C \geq 0$ such that

$$\left(\sum_{j=1}^m \|T(x_j^{(1)}, \dots, x_j^{(n)})\|^p \right)^{1/p} \leq C \left(\sup_{(\varphi_1, \dots, \varphi_n) \in B_{E_1^*} \times \dots \times B_{E_n^*}} \sum_{j=1}^m |\varphi_1(x_j^{(1)}) \dots \varphi_n(x_j^{(n)})|^p \right)^{1/p}$$

for every $m \in \mathbb{N}$, $x_j^{(l)} \in E_l$ with $l = 1, \dots, n$ and $j = 1, \dots, m$.

This ideal goes back to the research report [1] of R. Alencar and M.C. Matos and was explored in [26]. As in the case of p -dominated multilinear operators, a Pietsch

Domination theorem is valid in this context (for a proof we mention [26], although the result is inspired by the case $p = 1$ from Alencar-Matos paper [1]; see also [23] for a recent general argument):

Theorem 5.8 (Alencar, Matos, 1989 and Çaliskan, Pellegrino, 2007). *$T \in \mathcal{L}(E_1, \dots, E_n; F)$ is p -semi-integral if and only if there exist $C \geq 0$ and a regular probability measure μ on the Borel σ -algebra $\mathcal{B}(B_{E_1^*} \times \dots \times B_{E_n^*})$ of $B_{E_1^*} \times \dots \times B_{E_n^*}$ endowed with the product of the weak star topologies $\sigma(E_l^*, E_l)$, $l = 1, \dots, n$, such that*

$$\|T(x_1, \dots, x_n)\| \leq C \left(\int_{B_{E_1^*} \times \dots \times B_{E_n^*}} |\varphi_1(x_1) \dots \varphi_n(x_n)|^p d\mu(\varphi_1, \dots, \varphi_n) \right)^{1/p}$$

Corollary 5.9. *If $1 \leq p \leq q < \infty$, then $\mathcal{L}_{si,p} \subset \mathcal{L}_{si,q}$.*

It is well-known that, as it happens with the ideal of p -dominated multilinear operators, this ideal satisfies a Dvoretzky-Rogers type theorem.

This “size” of this class is strongly connected to the “size” of the class of p -dominated multilinear operators. For example, in [26] it is shown that

$$(5.1) \quad \mathcal{L}_{si,p}(E_1, \dots, E_n; F) \subset \mathcal{L}_{d,np}(E_1, \dots, E_n; F).$$

In fact, if $T \in \mathcal{L}_{si,p}(E_1, \dots, E_n; F)$ then

$$\begin{aligned} \left(\sum_{j=1}^{\infty} \|T(x_1^{(j)}, \dots, x_n^{(j)})\|^p \right)^{1/p} &\leq C \left(\sup_{\varphi_l \in B_{E_l^*}, l=1, \dots, n} \sum_{j=1}^{\infty} |\varphi_1(x_1^{(j)}) \dots \varphi_n(x_n^{(j)})|^p \right)^{1/p} \\ &\leq C \sup_{\varphi_l \in B_{E_l^*}, l=1, \dots, n} \left(\sum_{j=1}^{\infty} |\varphi_1(x_1^{(j)})|^{np} \right)^{\frac{1}{np}} \dots \left(\sum_{j=1}^{\infty} |\varphi_n(x_n^{(j)})|^{np} \right)^{\frac{1}{np}} \\ &= C \left\| (x_1^{(j)})_{j=1}^{\infty} \right\|_{w,np} \dots \left\| (x_n^{(j)})_{j=1}^{\infty} \right\|_{w,np}. \end{aligned}$$

In view of the “small size” of the class of p -dominated multilinear operators, the inclusion (5.1) might be viewed as a bad property.

5.3. Strongly summing multilinear operators. If $p \geq 1$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is strongly p -summing ($T \in \mathcal{L}_{ss,p}(E_1, \dots, E_n; F)$) if there exists a constant $C \geq 0$ such that

$$(5.2) \quad \left(\sum_{j=1}^m \|T(x_j^{(1)}, \dots, x_j^{(n)})\|^p \right)^{1/p} \leq C \left(\sup_{\phi \in B_{\mathcal{L}(E_1, \dots, E_n; \mathbb{K})}} \sum_{j=1}^m |\phi(x_j^{(1)}, \dots, x_j^{(n)})|^p \right)^{1/p}.$$

for every $m \in \mathbb{N}$, $x_j^{(l)} \in E_l$ with $l = 1, \dots, n$ and $j = 1, \dots, m$.

The multi-ideal of strongly p -summing multilinear operators is due to V. Dimant [38] is perhaps the class that best translates to the multilinear setting the properties of the original linear concept. For example, a Grothendieck type theorem and a Pietsch-Domination type theorem are valid:

Theorem 5.10 (Dimant, 2003). ([38]) *Every $T \in \mathcal{L}(^n \ell_1; \ell_2)$ is strongly 1-summing.*

Theorem 5.11 (Dimant, 2003). ([38]) *$T \in \mathcal{L}(E_1, \dots, E_n; F)$ is strongly p -summing if, and only if, there are a probability measure μ on $B_{(E_1 \otimes \pi \dots \otimes \pi E_n)^*}$, with the weak-star topology, and a constant $C \geq 0$ so that*

$$(5.3) \quad \|T(x_1, \dots, x_n)\| \leq C \left(\int_{B_{(E_1 \otimes \pi \dots \otimes \pi E_n)^*}} |\varphi(x_1 \otimes \dots \otimes x_n)|^p d\mu(\varphi) \right)^{\frac{1}{p}}$$

for all $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$,

The following intriguing result shows that in special situations the class of strongly p -summing multilinear maps contains the ideal of multiple p -summing operators:

Theorem 5.12 (Mezrag, Saadi, 2009). ([64]) *Let $1 < p < \infty$. If E_j is an \mathcal{L}_p -space for all $j = 1, \dots, n$ and F is an \mathcal{L}_{p^*} -space, then*

$$\mathcal{L}_{m,p^*}(E_1, \dots, E_n; F) \subset \mathcal{L}_{ss,p^*}(E_1, \dots, E_n; F).$$

It is not hard to prove that a Dvoretzky-Rogers Theorem is also valid for this class. Besides, the class has a nice size in the sense that no coincidence theorem can hold for n -linear maps if there is no analogue for linear operators. This indicates that this class is not “unnecessarily big”.

5.4. Absolutely summing multilinear operators. If $\frac{1}{p} \leq \frac{1}{q_1} + \dots + \frac{1}{q_n}$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is absolutely $(p; q_1, \dots, q_n)$ -summing at the point $a = (a_1, \dots, a_n) \in E_1 \times \dots \times E_n$ when

$$\left(T(a_1 + x_j^{(1)}, \dots, a_n + x_j^{(n)}) - T(a_1, \dots, a_n) \right)_{j=1}^{\infty} \in \ell_p(F)$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{q_k}^w(E_k)$. This class is denoted by $\mathcal{L}_{as,(p;q_1,\dots,q_n)}^{(a)}$. When a is the origin call simply absolutely $(p; q_1, \dots, q_n)$ -summing and represent by $\mathcal{L}_{as,(p;q_1,\dots,q_n)}$ (when $q_1 = \dots = q_n = q$ we write $\mathcal{L}_{as,(p;q)}$ and when $q_1 = \dots = q_n = q = p$ we just write $\mathcal{L}_{as,p}$). In the case that T is absolutely $(p; q_1, \dots, q_n)$ -summing at every $(a_1, \dots, a_n) \in E_1 \times \dots \times E_n$ we say that T is absolutely p -summing everywhere and we write $T \in \mathcal{L}_{as,(p;q_1,\dots,q_n)}^{ev}(E_1, \dots, E_n; F)$ (when $q_1 = \dots = q_n = q$ we write $\mathcal{L}_{as,(p;q)}^{ev}$ and when $q_1 = \dots = q_n = q = p$ we just write $\mathcal{L}_{as,p}^{ev}$).

The class of absolutely $(p; q_1, \dots, q_n)$ -summing operators (when $a = 0$) seems to have appeared for the first time in [1]. The starting point of the theory of absolutely summing is perhaps the result due to A. Defant and J. Voigt (see [1]), known as Defant-Voigt Theorem, which asserts that every continuous multilinear form is $(1; 1, \dots, 1)$ -summing. We prove here a slightly more general version which can be found in ([18]):

Theorem 5.13 (The generalized Defant-Voigt Theorem, 2007). *Let $A \in \mathcal{L}(E_1, \dots, E_n; F)$ and suppose that there exist $1 \leq r < n$ and $C > 0$ so that for any $x_1 \in E_1, \dots, x_r \in E_r$, the s -linear ($s = n - r$) mapping $A_{x_1 \dots x_r}(x_{r+1}, \dots, x_n) = A(x_1, \dots, x_n)$ is absolutely $(p; q_1, \dots, q_s)$ -summing and*

$$\|A_{x_1 \dots x_r}\|_{as(p;q_1,\dots,q_s)} \leq C \|A\| \|x_1\| \dots \|x_r\|.$$

Then A is absolutely $(p; 1, \dots, 1, q_1, \dots, q_s)$ -summing. In particular

$$\mathcal{L}(E_1, \dots, E_n; \mathbb{K}) = \mathcal{L}_{as,1}(E_1, \dots, E_n; \mathbb{K})$$

Proof. (Sketch) Given $m \in \mathbb{N}$ and $x_1^{(1)}, \dots, x_1^{(m)} \in E_1, \dots, x_n^{(1)}, \dots, x_n^{(m)} \in E_n$, let us consider $\varphi_j \in B_{F'}$ such that $\left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\| = \varphi_j(A(x_1^{(j)}, \dots, x_n^{(j)}))$ for every $j = 1, \dots, m$. Fix $b_1, \dots, b_m \in \mathbb{K}$ so that $\sum_{j=1}^m |b_j|^q = 1$, where $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\left(\sum_{j=1}^m \left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\|^p \right)^{\frac{1}{p}} = \left\| \left(\left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\| \right)_{j=1}^m \right\|_p = \sum_{j=1}^m b_j \left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\|.$$

If λ is the Lebesgue measure on $I = [0, 1]^r$, we have

$$\begin{aligned} & \int_I \sum_{j=1}^m \left(\prod_{l=1}^r r_j(t_l) \right) b_j \varphi_j A \left(\sum_{j_1=1}^m r_{j_1}(t_1) x_1^{(j_1)}, \dots, \sum_{j_r=1}^m r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \dots, x_n^{(j)} \right) d\lambda \\ &= \sum_{j, j_1, \dots, j_r=1}^m b_j \varphi_j A(x_1^{(j_1)}, \dots, x_r^{(j_r)}, x_{r+1}^{(j)}, \dots, x_n^{(j)}) \int_0^1 r_j(t_1) r_{j_1}(t_1) dt_1 \dots \int_0^1 r_j(t_r) r_{j_r}(t_r) dt_r \\ &= \sum_{j=1}^m \sum_{j_1=1}^m \dots \sum_{j_r=1}^m b_j \varphi_j A(x_1^{(j_1)}, \dots, x_r^{(j_r)}, x_{r+1}^{(j)}, \dots, x_n^{(j)}) \delta_{jj_1} \dots \delta_{jj_r} = \sum_{j=1}^m b_j \varphi_j A(x_1^{(j)}, \dots, x_n^{(j)}). \end{aligned}$$

For $z_l = \sum_{j=1}^m r_j(t_l) x_l^{(j)}$, $l = 1, \dots, r$, we get

$$\begin{aligned} & \left(\sum_{j=1}^m \left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\|^p \right)^{\frac{1}{p}} = \sum_{j=1}^m b_j \varphi_j A(x_1^{(j)}, \dots, x_n^{(j)}) \\ & \leq \int_I \left| \sum_{j=1}^m \left(\prod_{l=1}^r r_j(t_l) \right) b_j \varphi_j A \left(\sum_{j_1=1}^m r_{j_1}(t_1) x_1^{(j_1)}, \dots, \sum_{j_r=1}^m r_{j_r}(t_r) x_r^{(j_r)}, x_{r+1}^{(j)}, \dots, x_n^{(j)} \right) \right| d\lambda \end{aligned}$$

and after standard calculations we get

$$\left(\sum_{j=1}^m \left\| A(x_1^{(j)}, \dots, x_n^{(j)}) \right\|^p \right)^{\frac{1}{p}} \leq C \|A\| \left(\prod_{l=1}^r \left\| (x_l^{(j)})_{j=1}^m \right\|_{w,1} \right) \left(\prod_{l=r+1}^n \left\| (x_l^{(j)})_{j=1}^m \right\|_{w,q_l} \right).$$

□

Using a generalized version of Grothendieck's Inequality, D. Pérez García proved a striking generalization of Theorem 5.5:

Theorem 5.14 (Pérez-García, 2002). ([72, 75]) $\mathcal{L}_{as,(1,2)}({}^n c_0; \mathbb{K}) = \mathcal{L}({}^n c_0; \mathbb{K})$ for every $n \geq 2$.

A recent result from Blasco et al [5] shows that the crucial cases of Theorem 5.14 are precisely the cases $n = 2$ and $n = 3$:

Theorem 5.15 (Blasco, Botelho, Pellegrino, Rueda, 2010). *Let $1 \leq r \leq 2$. If $\mathcal{L}(^2E; \mathbb{K}) = \mathcal{L}_{as,(1,r)}(^2E; \mathbb{K})$ and $\mathcal{L}(^3E; \mathbb{K}) = \mathcal{L}_{as,(1,r)}(^3E; \mathbb{K})$, then*

$$\mathcal{L}(^nE; \mathbb{K}) = \mathcal{L}_{as,(1,r)}(^nE; \mathbb{K})$$

for every $n \geq 2$.

Proof. (Sketch of the proof when n is odd) Induction: Suppose that the result is valid for a fixed odd k and we shall prove that it is also true for $k+2$. Let $T \in \mathcal{L}(^{k+2}E; \mathbb{K})$ and consider

$$F = E \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E \text{ (} k \text{ times)}$$

$$G = E \hat{\otimes}_\pi E.$$

Consider a bilinear form

$$B \in \mathcal{L}(F, G; \mathbb{K})$$

so that

$$B(x^1 \otimes \cdots \otimes x^k, x^{k+1} \otimes x^{k+2}) = T(x^1, \dots, x^{k+2}).$$

Let $x_j^{(s)} \in E$ for $j = 1, \dots, m$ and $s = 1, \dots, k+2$. Using Defant-Voigt Theorem for B and the induction hypothesis one can find a constant C so that

$$\begin{aligned} & \sum_{j=1}^m \left| T(x_j^{(1)}, \dots, x_j^{(k+2)}) \right| \\ &= \sum_{j=1}^m \left| B(x_j^{(1)} \otimes \cdots \otimes x_j^{(k)}, x_j^{(k+1)} \otimes x_j^{(k+2)}) \right| \\ &\leq \pi_{(1;1)}(B) \left\| (x_j^{(1)} \otimes \cdots \otimes x_j^{(k)})_{j=1}^m \right\|_{\ell_1^w(E \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E)} \left\| (x_j^{(k+1)} \otimes x_j^{(k+2)})_{j=1}^m \right\|_{\ell_1^w(E \hat{\otimes}_\pi E)} \\ &\leq C \|B\| \left\| (x_j^{(1)})_{j=1}^m \right\|_{\ell_r^w(E)} \cdots \left\| (x_j^{(k+2)})_{j=1}^m \right\|_{\ell_r^w(E)} \end{aligned}$$

and the proof is done. \square

For general Banach spaces, the class $\mathcal{L}_{as,(1,2)}(^nE; \mathbb{K})$ also plays an important role, as an “upper bound” for the classes of p -dominated multilinear mappings [42, 73]:

Theorem 5.16 (Floret, Matos, 1995 (complex case), Pérez-García, 2003). *Let $n \in \mathbb{N}$, $n \geq 2$ and $p \geq 1$. If E is a Banach space, then*

$$\mathcal{L}_{d,p}(^nE; \mathbb{K}) \subset \mathcal{L}_{as,(1,2)}(^nE; \mathbb{K}).$$

At a first glance the concept of absolutely summing multilinear operator seems to be the natural multilinear definition of absolute summability. However it is easy to find bad properties which makes the ideal very different from the linear ideal.

For example, no general Inclusion Theorem is valid. In fact, Defant-Voigt Theorem ensures that

$$\mathcal{L}_{as,1}(^2\ell_2; \mathbb{K}) = \mathcal{L}(^n\ell_2; \mathbb{K})$$

but it is easy to show that

$$\mathcal{L}_{as,2}({}^2\ell_2; \mathbb{K}) \neq \mathcal{L}({}^2\ell_2; \mathbb{K}).$$

Besides, contrary to the linear case, several coincidence theorems hold, and this behavior removes the linear essence from this class. For example, Grothendieck's Theorem is valid but there are several other coincidence situations with absolutely no linear analogue, as

$$(5.4) \quad \mathcal{L}_{as,1}({}^n\ell_2; F) = \mathcal{L}({}^n\ell_2; F)$$

for all $n \geq 2$ and all F . Since (5.4) is not true for $n = 1$, from now on we call coincidence situations as (5.4) by "artificial coincidence situation".

Moreover, the polynomial version of this class is not an holomorphy type (this is a bad property!) and, in the terminology of [27], this bad property is reinforced since this class is not compatible with the linear ideal of absolutely summing operators. Despite its bad properties, this class has some challenging problems (see, for example, [14, 48, 67]).

As it occurs for multiple summing multilinear operators, in ([14]) it was shown that a full Inclusion Theorem is valid when the spaces from the domain are \mathcal{L}_∞ -spaces:

Theorem 5.17 (Botelho, Michels, Pellegrino, 2010). *Let $1 \leq p \leq q \leq \infty$ and E_1, \dots, E_n be \mathcal{L}_∞ -spaces. Then*

$$\mathcal{L}_{as,p}(E_1, \dots, E_n; F) \subset \mathcal{L}_{as,q}(E_1, \dots, E_n; F).$$

In some cases, surprisingly, the inclusion theorem holds in the opposite direction than the expected [48] (i.e. if p increases, the ideal decreases):

Theorem 5.18 (Junek, Matos, Pellegrino, 2008). *If E has cotype 2, F is any Banach space and $n \geq 2$, then*

$$\mathcal{L}_{as,q}({}^nE; F) \subset \mathcal{L}_{as,p}({}^nE; F)$$

holds true for $1 \leq p \leq q \leq 2$.

The class of everywhere absolutely p -summing multilinear operators was introduced by M.C. Matos [59] but he credits the idea to Richard Aron. It is easy to show that $\mathcal{L}_{m,p} \subset \mathcal{L}_{as,p}^{ev}$ and, as it occurs for $\mathcal{L}_{ss,p}$ and $\mathcal{L}_{m,p}$, this class has no artificial coincidence theorem (a proof can be found in [71]):

Proposition 5.19. *If $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{as,(q;p_1, \dots, p_n)}^{ev}(E_1, \dots, E_n; F)$, then*

$$\mathcal{L}(E_j; F) = \Pi_{q,p_j}(E_j; F), j = 1, \dots, n.$$

5.5. Strongly multiple summing multilinear operators: the last attempt. If $p \geq 1$, $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is strongly multiple p -summing ($T \in \mathcal{L}_{sm,p}(E_1, \dots, E_n; F)$) if there exists $C \geq 0$ such that

$$(5.5) \quad \left(\sum_{j_1, \dots, j_n=1}^m \|T(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)})\|^p \right)^{1/p} \leq C \left(\sup_{\phi \in B_{\mathcal{L}(E_1, \dots, E_n; \mathbb{K})}} \sum_{j_1, \dots, j_n=1}^m |\phi(x_{j_1}^{(1)}, \dots, x_{j_n}^{(n)})|^p \right)^{1/p}$$

for every $m \in \mathbb{N}$, $x_{j_l}^{(l)} \in E_l$ with $l = 1, \dots, n$ and $j_l = 1, \dots, m$.

The multi-ideal of strongly multiple p -summing multilinear operators was introduced in [18] and has not been explored since then. It contains the ideals $\mathcal{L}_{ss,p}$ and $\mathcal{L}_{m,p}$. All nice properties from $\mathcal{L}_{ss,p}$ are also valid except perhaps for versions of Pietsch Domination Theorem (and inclusion theorem) which are unknown. The size of this class is potentially better than the sizes of $\mathcal{L}_{ss,p}$ and $\mathcal{L}_{m,p}$ since despite containing these two classes, it is also known that for this class no coincidence theorem can hold for n -linear maps if there is no analogue for linear operators. So, even having a nice size, this class has no artificial coincidence results, with no linear analogue.

6. DESIRED PROPERTIES FOR A NICE MULTI-IDEAL EXTENSION OF ABSOLUTELY SUMMING OPERATORS

In [18, 26] and it is shown that

$$\begin{aligned}\mathcal{L}_{d,p} &\subset \mathcal{L}_{si,p} \subset \mathcal{L}_{m,p} \subset \mathcal{L}_{as,p}^{ev} \subset \mathcal{L}_{as,p}. \\ \mathcal{L}_{d,p} &\subset \mathcal{L}_{si,p} \subset \mathcal{L}_{ss,p} \subset \mathcal{L}_{sm,p}. \\ \mathcal{L}_{d,p} &\subset \mathcal{L}_{si,p} \subset \mathcal{L}_{m,p} \subset \mathcal{L}_{sm,p}.\end{aligned}$$

It is not difficult to show that $\mathcal{L}_{d,p}$ is cud, csm and it is well known that the Dvoretzky-Rogers Theorem is true, and also a Pietsch Domination Theorem (and, of course, the inclusion theorem) holds. On the other hand, as we have mentioned before this class is small and the Grothendieck Theorem is not true. As the above table shows the class $\mathcal{L}_{sm,p}$ is much bigger and from [18] we know that this class is cud, csm, and Dvoretzky-Rogers Theorem and Grothendieck's ℓ_1 - ℓ_2 Theorem are valid. More generally, this class contains the better-known class of multiple summing multilinear operators and hence it inherits all the known coincidence theorems for the class of multiple summing operators. In some sense, it is natural to expect that all reasonable multilinear extensions $\mathcal{M} = (\mathcal{M}_p)_{p \geq 1}$ of the ideal of absolutely summing operators should satisfy $\mathcal{L}_{d,p} \subset \mathcal{M}_p \subset \mathcal{L}_{sm,p}$.

Below we list the properties of each class:

Property/ Class	$\mathcal{L}_{d,p}$	$\mathcal{L}_{si,p}$	$\mathcal{L}_{ss,p}$	$\mathcal{L}_{m,p}$	$\mathcal{L}_{sm,p}$	$\mathcal{L}_{as,p}$	$\mathcal{L}_{as,p}^{ev}$
cud	Yes	Yes	Yes	Yes	Yes	No	Yes
csm	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Grothendieck Theorem	No	No	Yes	Yes	Yes	Yes	Yes
Inclusion Theorem	Yes	Yes	Yes	No	?	No	No
Dvoretzky-Rogers Theorem	Yes	Yes	Yes	Yes	Yes	No	Yes
$\mathcal{L}_{d,p} \subset \cdot \subset \mathcal{L}_{sm,p}$	Yes	Yes	Yes	Yes	Yes	No	?

Taking into account the main properties of the linear ideal of absolutely summing operators, we propose the following concept of “desired generalization of $(\Pi_p)_{p \geq 1}$ ”:

Definition 6.1. *A family of normed ideals of multilinear mappings $(\mathcal{M}_p)_{p \geq 1}$ is a desired generalization of $(\Pi_p)_{p \geq 1}$ if*

- (i) $\mathcal{L}_{d,p} \subset \mathcal{M}_p \subset \mathcal{L}_{sm,p}$ for all p and the inclusions have norm ≤ 1 .
- (ii) \mathcal{M}_p is csm for all p .
- (iii) \mathcal{M}_p is cud for all p .

- (iv) $\mathcal{M}_p \subset \mathcal{M}_q$ whenever $p \leq q$.
- (v) Grothendieck's Theorem and a Dvoretzky-Rogers theorem are valid.

First, observe that if $\mathcal{L}_{d,p} \subset \mathcal{M}_p \subset \mathcal{L}_{sm,p}$ then Dvoretzky-Rogers theorem is valid.

Note that, accordingly to the above table, $(\mathcal{L}_{ss,p})_{p \geq 1}$ is a desirable generalization of the family $(\Pi_p)_{p \geq 1}$. A desired generalization will be called “desired family”.

Definition 6.2. A desired family $\mathcal{M} = (\mathcal{M}_p)_{p \geq 1}$ is maximal if whenever $\mathcal{M}_p \subset \mathcal{I}_p$ for all p (and the inclusion has norm ≤ 1) and $(\mathcal{I}_p)_{p \geq 1}$ is a desired family, then $\mathcal{M}_p = \mathcal{I}_p$ for all p .

Similarly, a desired family $\mathcal{M} = (\mathcal{M}_p)_{p \geq 1}$ is minimal if whenever $\mathcal{I}_p \subset \mathcal{M}_p$ for all p (and the inclusion has norm ≤ 1) and $(\mathcal{I}_p)_{p \geq 1}$ is a desired family, then $\mathcal{M}_p = \mathcal{I}_p$ for all p .

Theorem 6.3. There exists a desired family of multilinear mappings which is maximal.

Proof. Let

$$D = \{ \mathcal{M}^\lambda = (\mathcal{M}_p^\lambda)_{p \geq 1} : \mathcal{M}^\lambda \text{ is a desired family for every } \lambda \in \Lambda \}.$$

In D we consider the partial order

$$(6.1) \quad \mathcal{M}^{\lambda_1} \leq \mathcal{M}^{\lambda_2} \Leftrightarrow \mathcal{M}_p^{\lambda_1} \subseteq \mathcal{M}_p^{\lambda_2} \text{ and } \|\cdot\|_{\mathcal{M}_p^{\lambda_2}} \leq \|\cdot\|_{\mathcal{M}_p^{\lambda_1}} \text{ for all } p \geq 1.$$

Note that $D \neq \emptyset$ since $(\mathcal{L}_{ss,p})_{p \geq 1} \in D$. We just need to show that Zorn's Lemma is applicable in order to yield the existence of a maximal family.

If $O \subset D$ is totally ordered and $\Lambda_O = \{ \lambda \in \Lambda : \mathcal{M}^\lambda = (\mathcal{M}_p^\lambda)_{p \geq 1} \in O \}$, consider the class

$$\mathcal{U} = (\mathcal{U}_p)_{p \geq 1},$$

where, for each $p \geq 1$, $\mathcal{U}_p = \bigcup_{\lambda \in \Lambda_O} \mathcal{M}_p^\lambda$.

In Λ_O we consider the direction

$$(6.2) \quad \lambda_1 \leq \lambda_2 \Leftrightarrow \mathcal{M}^{\lambda_1} \leq \mathcal{M}^{\lambda_2}$$

and, for each $p \geq 1$, define

$$(6.3) \quad \|T\|_{\mathcal{U}_p} := \lim_{\lambda \in \Lambda_O} \|T\|_{\mathcal{M}_p^\lambda}.$$

Note that the above limit exists in view of (6.1) and (6.2). It is not difficult to show that $(\mathcal{U}_p(E_1, \dots, E_n; F), \|\cdot\|_{\mathcal{U}_p})$ is a normed space, for each E_1, \dots, E_n, F . Moreover, $(\mathcal{U}_p, \|\cdot\|_{\mathcal{U}_p})$ is a normed ideal and one can quickly verify that $(\mathcal{U}_p, \|\cdot\|_{\mathcal{U}_p})_{p \geq 1}$ is a desired family. So $\mathcal{U} = (\mathcal{U}_p)_{p \geq 1} \in D$ and $\mathcal{U} \geq \mathcal{M}$ for all $\mathcal{M} \in D$; hence Zorn's Lemma yields that D has a maximal element. \square

We also have:

Theorem 6.4. There exists a desired family of multilinear mappings which is minimal.

Proof. Consider the set D as in the proof of the above theorem and the partial order

$$\mathcal{M}^{\lambda_2} \leq \mathcal{M}^{\lambda_1} \Leftrightarrow \mathcal{M}_p^{\lambda_1} \subseteq \mathcal{M}_p^{\lambda_2} \text{ and } \|\cdot\|_{\mathcal{M}_p^{\lambda_2}} \leq \|\cdot\|_{\mathcal{M}_p^{\lambda_1}} \text{ for all } p \geq 1.$$

Let also $O \subset D$ and Λ_O be as before. Define

$$\mathcal{I} = (\mathcal{I}_p)_{p \geq 1}$$

where, for all $p \geq 1$, $\mathcal{I}_p = \bigcap_{\lambda \in \Lambda_O} \mathcal{M}_p^\lambda$. Note that, for all $p \geq 1$, if $T \in \mathcal{I}_p(E_1, \dots, E_n; F)$, then

$$(6.4) \quad \|T\|_{\mathcal{I}_p} = \lim_{\lambda \in \Lambda_O} \|T\|_{\mathcal{M}_p^\lambda}$$

defines a norm in $\mathcal{I}_p(E_1, \dots, E_n; F)$. In fact, from our hypotheses, for each $p \geq 1$, the inclusion

$$inc : \mathcal{L}_{d,p}(E_1, \dots, E_n; F) \rightarrow \mathcal{M}_p^\lambda(E_1, \dots, E_n; F)$$

has norm ≤ 1 for all $\lambda \in \Lambda_O$. So, it follows that $\{\|T\|_{\mathcal{M}_p^\lambda} : \lambda \in \Lambda_O\}$ is bounded from above by $\|T\|_{d,p}$, and so the limit in (6.4) exists. Moreover, for all $p \geq 1$, the inclusion

$$inc : \mathcal{M}_p^\lambda(E_1, \dots, E_n; F) \rightarrow \mathcal{L}_{sm,p}(E_1, \dots, E_n; F)$$

has norm ≤ 1 for every $\lambda \in \Lambda_O$. Hence

$$\|T\|_{\mathcal{M}_p} \geq \|T\|_{sm,p} \geq 0 \text{ for all } T \in \mathcal{M}_p^\lambda(E_1, \dots, E_n; F)$$

and so

$$\|T\|_{\mathcal{I}_p} = 0 \text{ if and only if } T = 0.$$

The other properties are easily verified and hence $(\mathcal{I}_p(E_1, \dots, E_n; F), \|\cdot\|_{\mathcal{I}_p})$ is a normed space for all E_1, \dots, E_n, F . The rest of the proof follows the lines of the previous proof. \square

7. OPEN PROBLEMS

From the previous section two open problems arise:

Problem 7.1. *Is $(\mathcal{L}_{sm,p})_{p \geq 1}$ a desired ideal? (we conjecture that it is not) If the answer is positive, it will be maximal.*

Problem 7.2. *Is $(\mathcal{L}_{ss,p})_{p \geq 1}$ a maximal or minimal desired ideal?*

The answer to the next problem seems to be “NO”, but to the best of our knowledge, it is an open problem:

Problem 7.3. *Is $(\mathcal{L}_{ss,p})_{p \geq 1} = (\mathcal{L}_{sm,p})_{p \geq 1}$?*

For the next problem that we will propose, we need to define a quite artificial multilinear version of absolutely summing operators.

Let n be a positive integer, $k \in \{1, \dots, n\}$ and $p \geq 1$. An n -linear operator $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is k -absolutely p -summing if there is a constant $C \geq 0$ so that

$$(7.1) \quad \left(\sum_{j_k=1}^{\infty} \|T(x_{j_1}, \dots, x_{j_k}, \dots, x_{j_n})\|^p \right)^{1/p} \leq C \|x_{j_1}\| \dots \left\| (x_{j_k})_{j_k=1}^{\infty} \right\|_{w,p} \dots \|x_{j_n}\|$$

for all $x_{j_i} \in E_i$ with $i = 1, \dots, k-1, k+1, \dots, n$ and all $(x_{j_k})_{j_k=1}^{\infty} \in l_p^w(E_k)$. In this case we write $T \in \mathcal{L}_{s,p}^k(E_1, \dots, E_n; F)$. The infimum of the C satisfying (7.1) defines a norm for $\mathcal{L}_{s,p}^k(E_1, \dots, E_n; F)$, denoted by $\|\cdot\|_{ks,p}$. These maps were essentially introduced in [18] as an example of an artificial generalization of absolutely summing operators.

Now, consider the new class, $\mathcal{L}_{qs,p}$, whose components we will call quasi-absolutely p -summing:

$$\mathcal{L}_{qs,p}(E_1, \dots, E_n; F) := \bigcap_{k=1}^n \mathcal{L}_{s,p}^k(E_1, \dots, E_n; F).$$

Defining

$$(7.2) \quad \|T\|_{qs,p} := \max_k \|T\|_{ks,p},$$

we get a norm for $\mathcal{L}_{qs,p}(E_1, \dots, E_n; F)$ and $(\mathcal{L}_{qs,p}, \|\cdot\|_{qs,p})$ is a Banach ideal.

The ideal $(\mathcal{L}_{qs,p}, \|\cdot\|_{qs,p})$ is not interesting since it is the linear ideal in a nonlinear disguise. So, it is interesting to show that this class is not a desired family. In order to do this it is necessary to answer the following problem:

Problem 7.4. *Is $(\mathcal{L}_{sm,p})_{p \geq 1} = (\mathcal{L}_{qs,p})_{p \geq 1}$?*

Note that it is plain that $\mathcal{L}_{sm,p} \subset \mathcal{L}_{qs,p}$ and the inclusion has norm ≤ 1 .

Any answer to the above problem will lead to very important conclusions:

- If the answer to the above problem is YES (we conjecture that this is not), then we have several serious bits of information: (i) the equality is nontrivial and the result will be interesting by its own; (ii) we conclude that $(\mathcal{L}_{sm,p})_{p \geq 1}$ is a (maximal) desired ideal and (the more important) we conclude that $(\mathcal{L}_{sm,p})_{p \geq 1}$ indeed possesses very nice properties. For example, besides the inclusion theorem (which was unknown for this class), since every linear coincidence situation $\Pi_p(E; F) = \mathcal{L}(E; F)$ is naturally extended to $\mathcal{L}_{qs,p}(^n E; F) = \mathcal{L}(^n E; F)$, so we will also have $\mathcal{L}_{sm,p}(^n E; F) = \mathcal{L}(^n E; F)$ and with all this information in hand, it would be natural to consider $(\mathcal{L}_{sm,p})_{p \geq 1}$ as the "perfect" generalization of $(\Pi_p)_{p \geq 1}$ than $(\mathcal{L}_{m,p})_{p \geq 1}$.

- If the answer to the above problem is NO (which we conjecture), then we conclude that $(\mathcal{L}_{qs,p})_{p \geq 1}$ is not a desired class, a reasonable situation, since the class $(\mathcal{L}_{qs,p})_{p \geq 1}$ is artificially constructed.

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(D. Pellegrino) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA PARABA, 58038-310 - JOÃO PESSOA, PARABA, BRAZIL, [J. Santos] DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE SERGIPE, 49.500-000 - ITABAIANA, SERGIPE, BRAZIL.